Computational Robotics

L1: Transformation matrix between vectors and frames

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Why Transformation Matrices?

In robotics, one or more **frames** are assigned to each link and to each object in the workcell. The transformation of frames is a fundamental concept enabling us to:

- compute the location (position & orientation) of robot links relative to each other;
- describe the position and orientation of objects in the workcell;
- specify the trajectory and velocity of the end-effector;
- describe and control forces during robot–environment interaction;
- implement sensory-based control (each sensor has its own reference frame).

Approach

We use **homogeneous transformations** – widely adopted in computer graphics and robotics.

Representation of a Point

Let (P_x, P_y, P_z) be the Cartesian coordinates of a point P in frame R_i.

The **homogeneous coordinates** of P w.r.t. R_i (with scaling factor $w = 1$):

$$ {}^{i}\mathrm{P} = \begin{bmatrix} {}^{i}P_x \\ {}^{i}P_y \\ {}^{i}P_z \\ 1 \end{bmatrix} \quad (4\times1 \text{ column vector}) \tag{2.1} $$

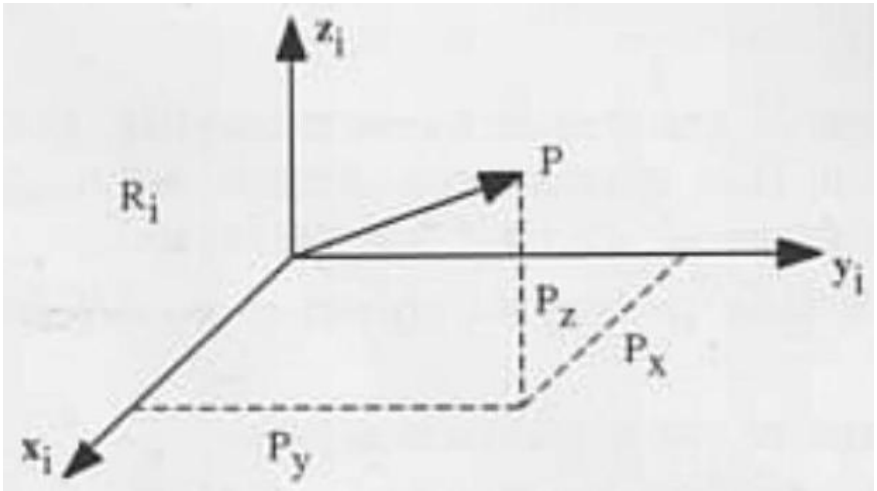


Fig. 2.1 – Point representation in frame R_i

Representation of a Direction and a Plane

Free Vector (Direction)

Fourth component is **zero** (vector at infinity):

$$ {}^{i}\mathsf{u} = \begin{bmatrix} {}^{i}u_x \\ {}^{i}u_y \\ {}^{i}u_z \\ 0 \end{bmatrix} \tag{2.2} $$

Plane

A plane Q with equation ${}^{i}\alpha x + {}^{i}\beta y + {}^{i}\gamma z + {}^{i}\delta = 0$ is represented by the **row vector**:

$$ {}^{i}\mathsf{Q} = \begin{bmatrix} {}^{i}\alpha & {}^{i}\beta & {}^{i}\gamma & {}^{i}\delta \end{bmatrix} \tag{2.3} $$

If point P lies on plane Q: ${}^{i}\mathsf{Q}\,{}^{i}\mathsf{P} = 0$.

Transformation of Frames

The transformation from frame R_i to frame R_j is represented by the (4×4) **homogeneous transformation matrix**:

$$ {}^{i}\mathrm{T}_{j} = \begin{bmatrix} {}^{i}\mathrm{R}_{j} & {}^{i}\mathrm{P}_{j} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^{i}\mathrm{s}_{j} & {}^{i}\mathrm{n}_{j} & {}^{i}\mathrm{a}_{j} & {}^{i}\mathrm{P}_{j} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.5} $$

where ${}^{i}\mathrm{s}_{j}$, ${}^{i}\mathrm{n}_{j}$, ${}^{i}\mathrm{a}_{j}$ are the unit vectors of x_j, y_j, z_j expressed in R_i, and ${}^{i}\mathrm{P}_{j}$ is the origin of R_j in R_i.

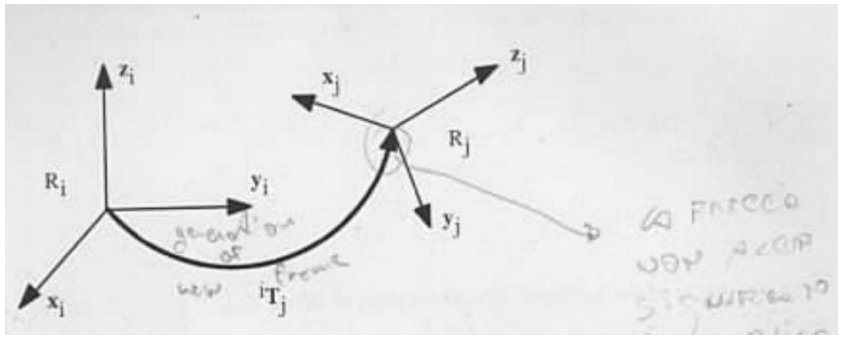


Fig. 2.2 – Transformation of frames

Transformation of Vectors

Let ${}^{j}\mathrm{P}$ be the homogeneous coordinates of point P in R_j.

The coordinates in R_i are obtained via:

$$ {}^{i}\mathrm{P} = {}^{i}\mathrm{T}_{j}\,{}^{j}\mathrm{P} \tag{2.6} $$

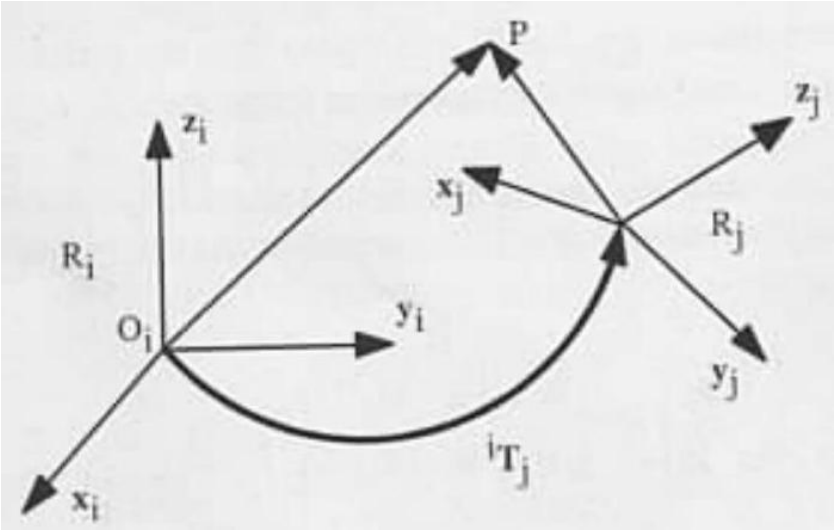


Fig. 2.3 – Transformation of a vector

Key meaning

${}^{i}\mathrm{T}_{j}$ converts coordinates expressed in R_j to coordinates expressed in R_i.

Example 2.1 – Reading ${}^{i}\mathrm{T}_{j}$ from a Figure

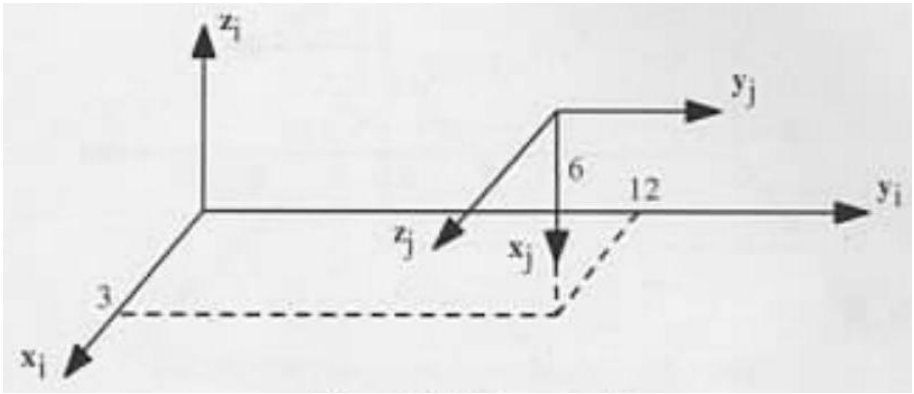


Fig. 2.4 – Example 2.1

$$ {}^{i}\mathrm{T}_{j} = \begin{bmatrix} 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 12 \\ -1 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{j}\mathrm{T}_{i} = \begin{bmatrix} 0 & 0 & -1 & 6 \\ 0 & 1 & 0 & -12 \\ 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Origin of R_j in R_i: $(3, 12, 6)$

Pure Translation

Trans(a, b, c) – translation along x, y, z by a, b, c:

$$\text{Trans}(a, b, c) = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.8)$$

Can be decomposed as
Trans(x, a) Trans(y, b) Trans(z, c) in any order.

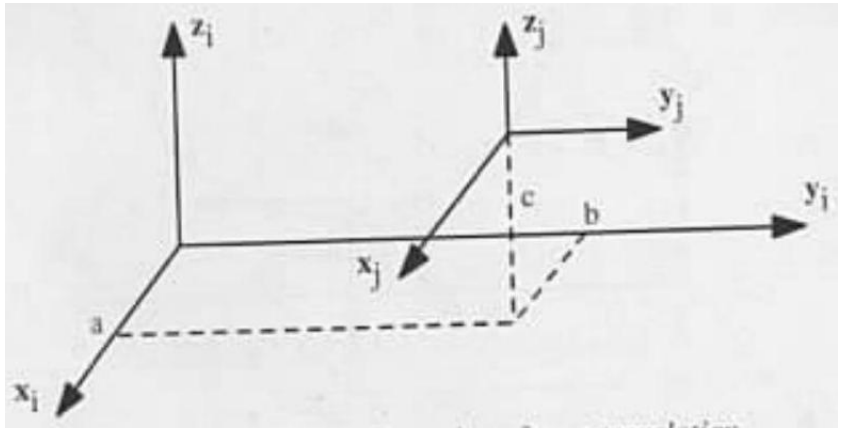


Fig. 2.5 – Pure translation

Rotations About Principal Axes

Rotation about x **by** θ**:**

$$\text{Rot}(x,\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\theta & -S\theta & 0 \\ 0 & S\theta & C\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.10)$$

Rotation about y **by** θ**:**

$$\text{Rot}(y,\theta) = \begin{bmatrix} C\theta & 0 & S\theta & 0 \\ 0 & 1 & 0 & 0 \\ -S\theta & 0 & C\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.11)$$

Rotation about z **by** θ**:**

$$\text{Rot}(z,\theta) = \begin{bmatrix} C\theta & -S\theta & 0 & 0 \\ S\theta & C\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.12)$$

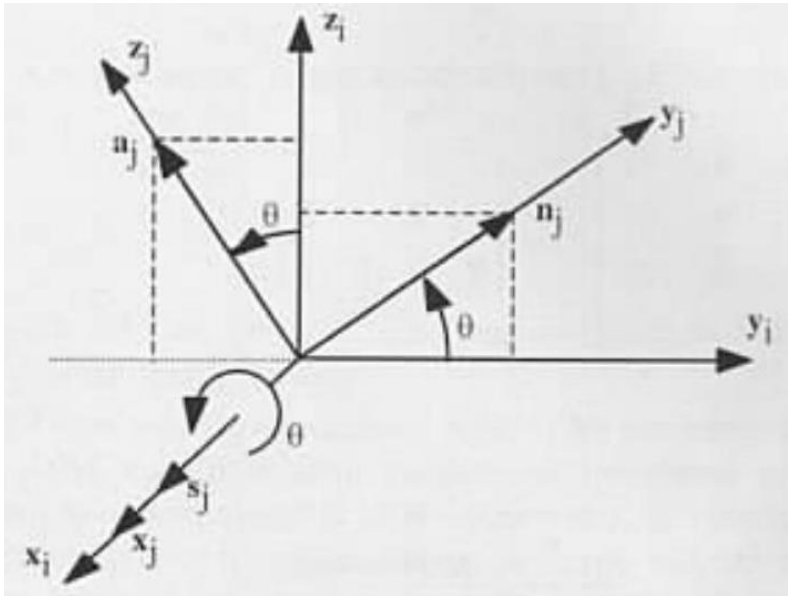


Fig. 2.6 – Rotation about x-axis

Key Properties (I)

Given $\mathrm{T} = \begin{bmatrix} \mathrm{R} & \mathrm{P} \\ 0 & 1 \end{bmatrix}$:

(a) R is the 3×3 direction cosine matrix; P is the translation. $\mathrm{R} = \mathrm{I}_3$ for pure translation; $\mathrm{P} = 0$ for pure rotation.

(b) R is **orthogonal**: $\mathrm{R}^{-1} = \mathrm{R}^T$. (2.14)

(c) Inverse: $({}^i\mathrm{T}_j)^{-1} = {}^j\mathrm{T}_i$

(d) $\mathrm{Rot}^{-1}(\mathrm{u}, \theta) = \mathrm{Rot}(\mathrm{u}, -\theta) = \mathrm{Rot}(-\mathrm{u}, \theta)$ (2.18)

(e) Explicit inverse formula:

$$\mathrm{T}^{-1} = \begin{bmatrix} \mathrm{R}^T & -\mathrm{R}^T\mathrm{P} \\ 0 & 1 \end{bmatrix} \tag{2.20}$$

Key Properties (II): Composition

(f) Product of two matrices

$$T_1 T_2 = \begin{bmatrix} R_1 R_2 & R_1 P_2 + P_1 \\ 0 & 1 \end{bmatrix} \tag{2.21}$$

Note: matrix multiplication is **non-commutative**: $T_1 T_2 \neq T_2 T_1$.

(g) k consecutive transformations w.r.t. current frame (right multiplication)

$$^0T_k = {}^0T_1 \, {}^1T_2 \, {}^2T_3 \cdots {}^{k-1}T_k \tag{2.22}$$

(h) Transformation defined w.r.t. reference frame (left multiplication)

$$^iT_{j'} = T \, {}^iT_j$$

Pre- vs. Post-Multiplication Rule

Fundamental Rule

- **Post-multiplication** (right): transformation defined w.r.t. the *current* frame R_j
- **Pre-multiplication** (left): transformation defined w.r.t. the *reference* frame R_i

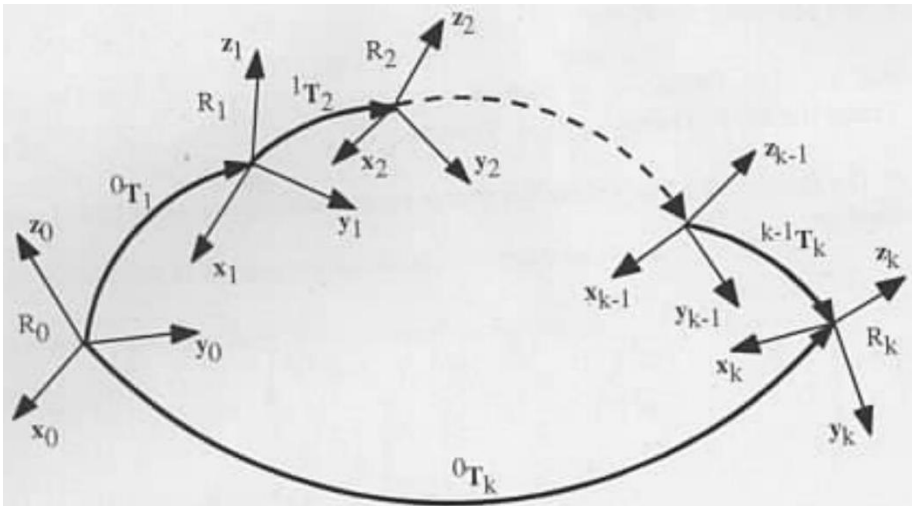


Fig. 2.7 – Right multiplication

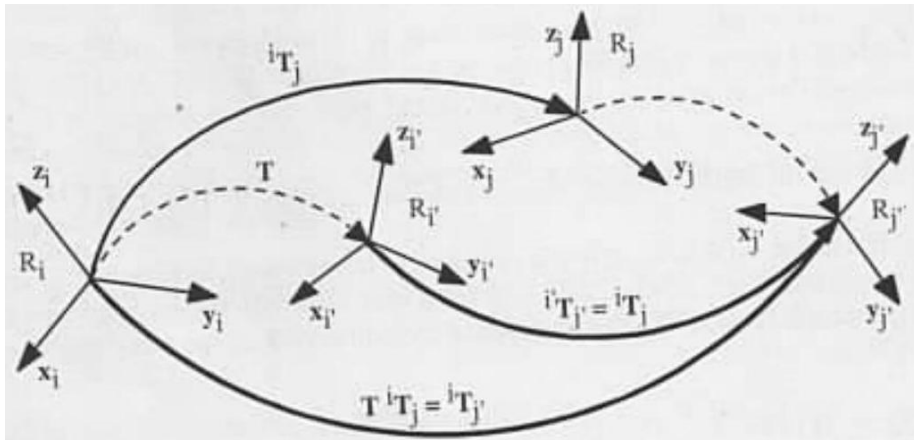


Fig. 2.8 – Left multiplication

Example 2.2 – Composite Transformation

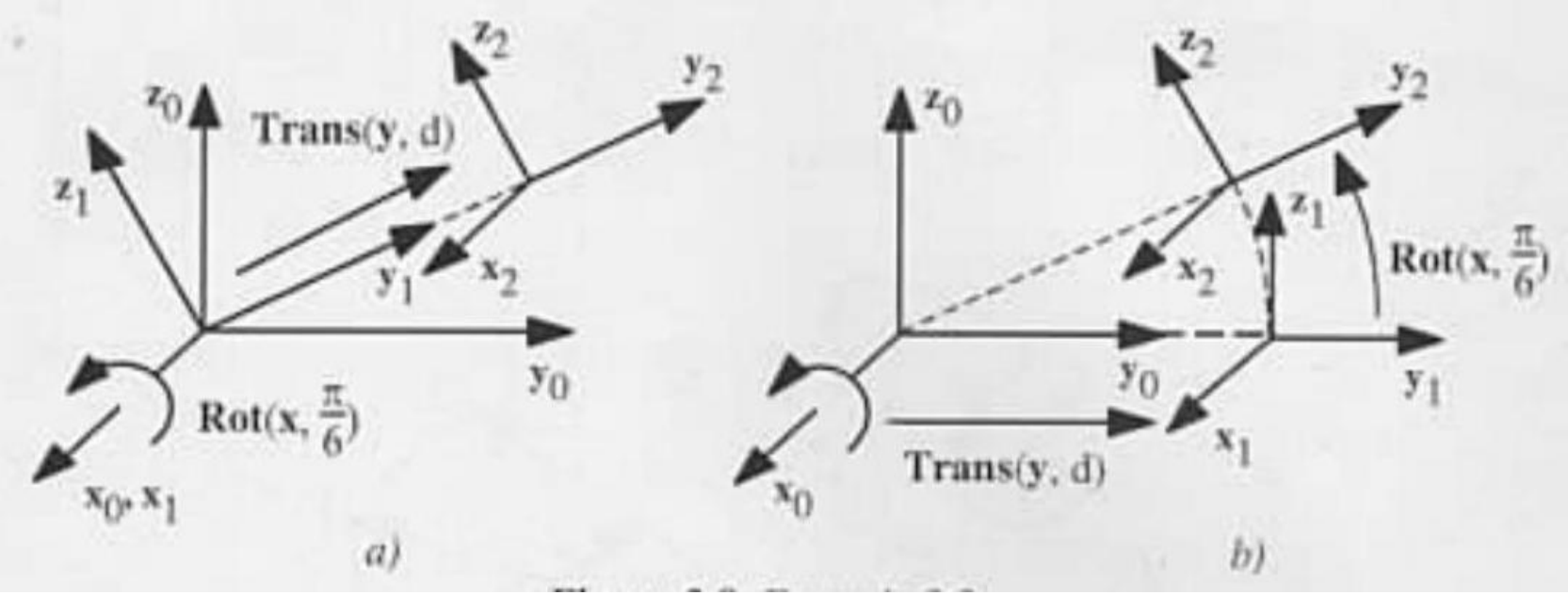


Fig. 2.9 – ${}^0T_2 = \text{Rot}(x, \pi/6)\ \text{Trans}(y, d)$

- **Left to right (a):** first apply $\text{Rot}(x, \pi/6)$ (current frame), then $\text{Trans}(y, d)$ (w.r.t. resulting frame).
- **Right to left (b):** first apply $\text{Trans}(y, d)$, then $\text{Rot}(x, \pi/6)$ (w.r.t. reference frame R_0).

Additional Properties

(i) Consecutive transformations about the same axis

$$\mathrm{Rot}(\mathrm{u}, \theta_1)\,\mathrm{Rot}(\mathrm{u}, \theta_2) = \mathrm{Rot}(\mathrm{u}, \theta_1 + \theta_2) \tag{2.23}$$

$$\mathrm{Rot}(\mathrm{u}, \theta)\,\mathrm{Trans}(\mathrm{u}, d) = \mathrm{Trans}(\mathrm{u}, d)\,\mathrm{Rot}(\mathrm{u}, \theta) \tag{2.24}$$

(j) Decomposition into translation and rotation

$$\mathrm{T} = \begin{bmatrix} \mathrm{I}_3 & \mathrm{P} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{R} & 0 \\ 0 & 1 \end{bmatrix} \tag{2.25}$$

Rotation About a General Axis (Rodrigues Formula)

Let $\mathrm{Rot}(\mathrm{u}, \theta)$ rotate by θ about unit vector $\mathrm{u} = [u_x, u_y, u_z]^T$ at the origin. The (3×3) rotation matrix is:

$$\mathrm{rot}(\mathrm{u}, \theta) = \mathrm{u}\mathrm{u}^T(1 - C\theta) + \mathrm{I}_3 C\theta + \hat{\mathrm{u}}\, S\theta \tag{2.31}$$

where the **skew-symmetric matrix** $\hat{\mathrm{u}}$ is:

$$\hat{\mathrm{u}} = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} \tag{2.32}$$

Note: $\mathrm{u} \times \mathrm{V} = \hat{\mathrm{u}}\mathrm{V}$.

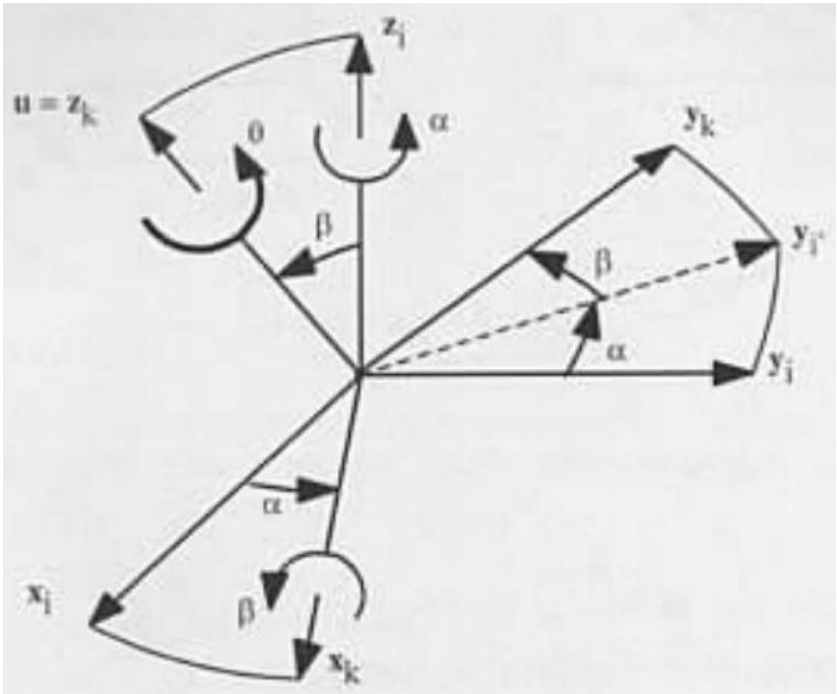


Fig. 2.10 – Rotation about general axis

Equivalent Angle and Axis of a General Rotation

Given an arbitrary rotation matrix T with elements $s_x, n_y, a_z, \ldots$, we can recover the equivalent angle θ and axis u:

$$C\theta = \frac{1}{2}(s_x + n_y + a_z - 1) \tag{2.34}$$

$$S\theta = \frac{1}{2}\sqrt{(n_z - a_y)^2 + (a_x - s_z)^2 + (s_y - n_x)^2} \tag{2.36}$$

$$\theta = \arctan(S\theta/C\theta), \quad 0 \leq \theta \leq \pi \tag{2.37}$$

Axis components (systems of these following eqs):

$$2u_x S\theta = n_z - a_y, \quad 2u_y S\theta = a_x - s_z, \quad 2u_z S\theta = s_y - n_x \tag{2.35}$$

Conclusion

Chapter Summary

- Introduced **homogeneous coordinates** for points, vectors, and planes.
- Developed the (4×4) **homogeneous transformation matrix** and its properties (orthogonality, inverse, composition, pre/post-multiplication rules).
- Derived rotation matrices about principal axes and about a **general axis** (Rodrigues formula).